ON THE CALCULATION OF COMPRESSIBILITY IN THE THEORY OF IDEALLY PLASTIC MEDIA

(OB UCHETE SZHIMAEMOSTI V TEORII IDEAL'NO PLASTICHESKIKH SRED)

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Irreversible deformations of a continuous medium may be accompanied by a change in volume. Below, we investigate the question of the calculation of compressibility in the theory of ideally plastic media. This leads to a generalization of the theorem of Mises [1] concerning the associated law for plastic flow in compressible media.

We remark that questions on the computation of the compressibility of ideally plastic media were examined in [2].

1. Consider an isotropic ideally plastic body which is acted upon by some loads. We denote the components of stress and strain by σ_{ij} and e_{ij}, respectively.

We assume that the following relations have been established as the results of experiments under homogeneous pressure:

$$\sigma = f(e) \quad (e = \varphi(\sigma)), \qquad \sigma = \frac{1}{3}\sigma_{ii}, \qquad e = \frac{1}{3}e_{ii} \tag{1.1}$$

We assume further that an irreversible change in the form of the medium occurs when the stresses attain a certain combination of values

$$\Phi(\sigma, \Sigma_2, \Sigma_3) = 0 \tag{1.2}$$

Here Σ_2, Σ_3 are the second and third invariants of the stress deviator.

An increment in the work of the stresses σ_{ij} on increments of the strain de_{ij} will have the form

$$dA = \sigma_{ij}\,de_{ij} = \sigma_{ij}\,de_{ij}' + 3\sigma de \tag{1.3}$$

where here and subsequently the upper prime is to be ascribed to the components of the stress deviator.

Following Mises [1] we look for an extremum of Expression (1.3), assuming that the deformed state is fixed and only the components of the stress tensor are varied. Taking into account (1.2), (1.1), we look for an extremum of the functional

$$dA = \sigma_{ij}\, de_{ij} - d\lambda_1\, \Phi(\sigma,\ \Sigma_2,\ \Sigma_3) - d\lambda_2(\sigma - f(e)) \tag{1.4}$$

From (1.4) we have

$$de_{ij}' + \delta_{ij}\, de = d\lambda_1 \left[\frac{\partial \Phi}{\partial \sigma}\frac{\partial \sigma}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial \Sigma_2}\frac{\partial \Sigma_2}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial \Sigma_3}\frac{\partial \Sigma_3}{\partial \sigma_{ij}}\right] + d\lambda_2 \frac{\partial \sigma}{\partial \sigma_{ij}} \tag{1.5}$$

From (1.5) follows

$$3de = d\lambda_1 \frac{d\Phi}{d\sigma} + d\lambda_2 \tag{1.6}$$

Eliminating the quantity $d\lambda_2$ from (1.5) and (1.6), we obtain

$$de_{ij}' = d\lambda_1 \left(\frac{\partial \Phi}{\partial \Sigma_2}\frac{\partial \Sigma_2}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial \Sigma_3}\frac{\partial \Sigma_3}{\partial \sigma_{ij}}\right) \tag{1.7}$$

Turning to the strain velocities we have

$$\varepsilon_{ij}' = \lambda \left(\frac{\partial \Phi}{\partial \Sigma_2}\frac{\partial \Sigma_2}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial \Sigma_3}\frac{\partial \Sigma_3}{\partial \sigma_{ij}}\right), \qquad \varepsilon_{ij} = \frac{de_{ij}}{dt},\ \lambda = \frac{d\lambda_1}{dt} \tag{1.8}$$

Hence, the following theorem may be stated. If, following Mises [1], we define the associated law of plastic flow from the representation of the extremality of an increment in the work of the stresses under the given state of deformation, then, for compressible ideally plastic media whose plasticity condition is of the form (1.2), the components of the strain velocity deviator are directly proportional to the partial derivatives with respect to the stress components of that part of the plasticity condition which depends on the second and third invariants of the stress deviator; furthermore, the expression for the associated law of plastic flow (1.8) does not at all depend on the compressibility law.

2. We examine, for example, the case of plane strain of an ideally plastic material under the plasticity condition

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4c^2, \qquad (c = \text{const}) \tag{2.1}$$

In this case let there exist the relation $\sigma = 3Ke$ (K = const). The problem is then statically determinate and the compressibility has no effect of the stress equations. It is well known that the stress equations are of hyperbolic type, with the equations for the characteristics having the form

$$\frac{dy}{dx} = \tan\left(\theta \pm \frac{\pi}{4}\right) \tag{2.2}$$

Along a characteristic there exists the relation

$$\omega \pm 2\theta = \mathrm{const} \left(\omega = \frac{1}{2c} (\sigma_x + \sigma_y)\right) \qquad (2.3)$$

Here θ is the angle formed by the first principal stress with the x-axis.

According to (1.7) and (1.8) the plastic flow law can be rewritten in the form

$$de_x = d\lambda_1 (\sigma_x - \sigma_y) + \frac{d\sigma}{3K}, \qquad de_y = d\lambda_1 (\sigma_y - \sigma_x) + \frac{d\sigma}{3K}$$
$$de_{xy} = 2 d\lambda_1 \tau_{xy} \qquad (2.4)$$

or

$$\varepsilon_x = \lambda (\sigma_x - \sigma_y) + \frac{\dot{\sigma}}{3K}, \qquad \varepsilon_y = \lambda (\sigma_y - \sigma_x) + \frac{\dot{\sigma}}{3K}$$
$$\varepsilon_{xy} = 2\lambda\tau_{xy} \qquad (2.5)$$

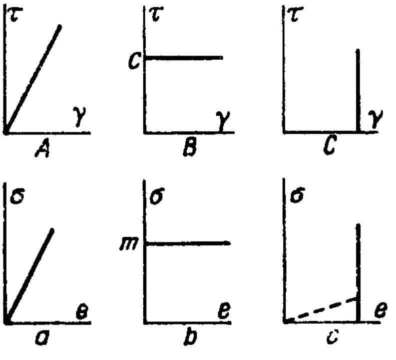
The upper dot denotes differentiation with respect to time.

If we pass to the components of the velocities of the displacements u and v along the x- and y-axes, then, eliminating the quantity λ from the relations (2.5), we obtain two equations of hyperbolic type whose characteristics coincide with (2.2). Along the characteristics there occur the relations

$$\tan\left(\theta \pm \frac{\pi}{4}\right) du - dv \pm \frac{2}{3K} \frac{\dot{\sigma}\, dx}{\cos 2\theta} = 0 \qquad (2.6)$$

or

$$dU - V d\theta + \frac{2\dot{\sigma}}{3K} \frac{\cos(\theta + \pi/4)}{\cos 2\theta} dx = 0$$
$$dV + U d\theta + \frac{2\dot{\sigma}}{3K} \frac{\cos(\theta - \pi/4)}{\cos 2\theta} dy = 0 \qquad (2.7)$$



Here U, V are the components of the displacement velocities along the characteristics.

3. We make a few remarks. First of all we note that if from the outset there are no restrictions on the compressibility, then the associated flow law will be of the form

$$\varepsilon_{ij} = \lambda \frac{\partial \Phi}{\partial \sigma_{ij}} \qquad (3.1)$$

As a consequence of (3.1) there occurs the "associated" compressibility of the material

$$3e = \lambda \frac{\partial \Phi}{\partial \sigma} \tag{3.2}$$

Generally speaking, the shear (deviator) components and the components characterizing the volume deformation may be assumed to be independent. The results of the simplest experiments are represented in the figure (it is clear that the indicated cases are far from all the possible representations). Shear stresses and strains are denoted by τ, γ, respectively; the mean stress and strain are denoted by σ and e respectively.

The diagrams *A, a* (Figure) correspond to a linear elastic body; the diagrams *B, b* correspond to an incompressible rigidly plastic body when $\varkappa \to \infty$.

A body with limited compressibility [3] corresponds to a combination of properties *A, c*. An ideal locking body [4,5] corresponds to a combination of *C, b* with $\varkappa \to \infty$.

It is likewise possible to investigate an elastic body which is capable of resisting volume deformation only up to a definite limit (combination *A, b*), etc.

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